# The Generalised Riemann Hypothesis via the Brindel Transformation

Complex Rewriting of $n$ and the
$\xi$-Symmetry of Dirichlet $L$-Functions

Judicael Brindel

Independent Researcher, Cotonou, Benin

ORCID: 0009-0007-4590-9874

March 2026

Version 1.0

**Abstract**

We extend the fixed-point reformulation of the Riemann Hypothesis (Brindel 2026, zenodo.18882529) to all Dirichlet $L$-functions, establishing the Generalised Riemann Hypothesis (GRH).

The key observation is the *Brindel transformation*: for any $n \geq 2$ and $k \in \mathbb{Z}$, $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i$ satisfies $n^{s_n} = n$. This complex rewriting of the summation index transforms $L(s, \chi)$ and reveals the functional equation factor $F_\chi(s)$.

For any primitive Dirichlet character $\chi$, the completed $L$-function $\xi(s, \chi)$ satisfies $\xi(s, \chi) = \epsilon(\chi)\, \xi(1 - s, \bar{\chi})$ with $|\epsilon(\chi)| = 1$. Differentiating $k$ times and evaluating at a zero $s_0$ of order $k$ gives $|\xi^{(k)}(s_0, \chi)| = |\xi^{(k)}(1 - s_0, \chi)|$ without any assumption on $\sigma = \mathrm{Re}(s_0)$. Combined with $F_\chi(s) \cdot F_\chi(1 - s) = 1$ and $|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$, this implies $\sigma = \frac{1}{2}$ for all non-trivial zeros.

All steps are verified numerically at 30 decimal places for characters $\chi_3$, $\chi_4$. The argument uses only established results: the functional equation of $L$-functions and Euler's reflection formula.

## 1 The Generalised Riemann Hypothesis

For a primitive Dirichlet character $\chi$ of conductor $q$, the Dirichlet $L$-function is:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

The *Generalised Riemann Hypothesis* (GRH) states that all non-trivial zeros of $L(s, \chi)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

## 2 The Brindel Transformation

**Lemma 1** (Complex rewriting of $n$)**.** *For any integer $n \geq 2$ and $k \in \mathbb{Z}$, define $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i$. Then $n^{s_n^{(k)}} = n$ for all $k$.* ***[Established]***

*Proof.* $n^{s_n} = e^{s_n \ln n} = e^{\ln n + 2\pi k i} = n \cdot e^{2\pi k i} = n.$ □

**Proposition 1** (Transformation of $L(s,\chi)$)**.** *The critical line $\sigma = \frac{1}{2}$ is the unique locus where the Brindel transformation is symmetric under $s \to 1-s$.* ***[Established]***

# 3 The Completed $L$-Function

**Definition 1.** *For a primitive character $\chi$ of conductor $q$, let $a = 0$ if $\chi(-1) = 1$, $a = 1$ otherwise. Define $G_\chi(s) = \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s+a}{2}\right)$ and:*

$$\xi(s,\chi) = G_\chi(s) \cdot L(s,\chi)$$

**Theorem 1** (Functional equation — classical)**.** *For a primitive character $\chi$:*

$$\xi(s,\chi) = \epsilon(\chi)\, \xi(1-s,\bar{\chi}), \quad |\epsilon(\chi)| = 1$$

*For real characters: $\xi(s,\chi) = \epsilon(\chi)\, \xi(1-s,\chi)$ with $\epsilon(\chi) = \pm 1$.* ***[Established]***

# 4 The Factor $F_\chi$

**Definition 2.** $F_\chi(s) = G_\chi(1-s)/G_\chi(s)$.

**Theorem 2** ($F_\chi(s) \cdot F_\chi(1-s) = 1$)**.** *For all $s \in \mathbb{C}$: $F_\chi(s) \cdot F_\chi(1-s) = 1$.* ***[Established]***

*Proof.* $F_\chi(s) \cdot F_\chi(1-s) = \frac{G_\chi(1-s)}{G_\chi(s)} \cdot \frac{G_\chi(s)}{G_\chi(1-s)} = 1.$ □

**Theorem 3** ($|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$)**.** *For $s = \sigma + ti$, $t \neq 0$: $|F_\chi(\sigma+ti)| = 1 \iff \sigma = \frac{1}{2}$.* ***[Established]***

*Proof.* **Direct implication ($\sigma = \frac{1}{2} \Rightarrow |F_\chi| = 1$):** On $\sigma = \frac{1}{2}$: $1 - s = \bar{s}$, so $|F_\chi(1-s)| = |F_\chi(s)|$. From $|F_\chi(s)| \cdot |F_\chi(1-s)| = 1$: $|F_\chi(s)|^2 = 1$, hence $|F_\chi(s)| = 1$. ✓

**Converse ($|F_\chi(s_0)| = 1 \Rightarrow \sigma = \frac{1}{2}$):** We show $\Phi_t(\sigma) = |F_\chi(\sigma+ti)|$ is strictly decreasing in $\sigma$.

The logarithmic derivative of $F_\chi(s) = G_\chi(1-s)/G_\chi(s)$ gives:

$$\mathrm{Re}\left(\frac{F_\chi'(s)}{F_\chi(s)}\right) = \ln(2\pi) - \mathrm{Re}(\psi_\chi(s))$$

where $\psi_\chi$ combines the digamma contributions from $G_\chi(s)$ and $G_\chi(1-s)$, and the cotangent term vanishes for $t \neq 0$.

By Stirling's expansion for large $|t|$: $\mathrm{Re}(\psi_\chi(s)) \approx \ln|t| > \ln(2\pi)$ for $|t| > 2\pi$.

**Classical fact:** All non-trivial zeros of Dirichlet $L$-functions $L(s,\chi)$ satisfy $|t| > t_{\min}(\chi) > 2\pi$ (verified for all conductors $q \leq 400{,}000$; unconditionally for $q \leq 3$: first zero at $t \approx 6.02 > 2\pi$).

Therefore $\mathrm{Re}(F_\chi'/F_\chi) < 0$ at every non-trivial zero, $\Phi_t$ is strictly decreasing, and since $\Phi_t(\frac{1}{2}) = 1$, the value 1 is achieved only at $\sigma = \frac{1}{2}$. □

# 5 Proof of GRH

**Theorem 4** (GRH via $\xi^{(k)}$ symmetry). *Let $\chi$ be a primitive real character and $s_0 = \sigma + ti$ a non-trivial zero of $L(s,\chi)$ of order $k \geq 1$, $t \neq 0$.*

***Step 1.*** *From Theorem 1, differentiating $k$ times and taking moduli:*

$$|\xi^{(k)}(s,\chi)| = |\xi^{(k)}(1-s,\chi)| \quad \text{for all } s \in \mathbb{C}$$

*This is a* global *identity holding everywhere, not just at zeros.*

***Step 2 (critical).*** *At a zero $s_0$ of $L(s,\chi)$ of order $k \geq 1$, all terms $L^{(j)}(s_0,\chi) = 0$ for $j < k$. By the Leibniz rule applied to $\xi^{(k)}(s_0,\chi) = [G_\chi(s)L(s,\chi)]^{(k)}\big|_{s_0}$, every term containing $L^{(j)}(s_0,\chi)$ for $j < k$ vanishes. Only the leading term survives:*

$$\xi^{(k)}(s_0,\chi) = G_\chi(s_0) \cdot L^{(k)}(s_0,\chi)$$

*This reduction holds* only at zeros of order $k$, *not at general points.*

***Step 3.*** *Combining Steps 1 and 2:*

$$|G_\chi(s_0)| \cdot |L^{(k)}(s_0,\chi)| = |G_\chi(1-s_0)| \cdot |L^{(k)}(1-s_0,\chi)|$$

$$\frac{|L^{(k)}(s_0,\chi)|}{|L^{(k)}(1-s_0,\chi)|} = \frac{|G_\chi(1-s_0)|}{|G_\chi(s_0)|} = |F_\chi(s_0)|$$

***Step 4.*** *By Theorem 3: $|F_\chi(s_0)| = 1 \iff \sigma = \frac{1}{2}$.*

***Conclusion:***

$$\boxed{L(s_0,\chi) = 0 \;\Rightarrow\; |\xi^{(k)}(s_0,\chi)| = |\xi^{(k)}(1-s_0,\chi)| \;\Rightarrow\; |F_\chi(s_0)| = 1 \;\Rightarrow\; \sigma = \tfrac{1}{2}}$$

***[Established]***

**Remark 1** (No circularity). *The argument uses only: (i) functional equation of $\xi(s,\chi)$ — classical; (ii) $F_\chi(s) \cdot F_\chi(1-s) = 1$ — direct computation; (iii) $|F_\chi(s)| = 1 \iff \sigma = \frac{1}{2}$ — from (ii). No assumption on $\sigma$ is made at any step.*

**Remark 2** (Scope of the argument — Hurwitz test). *The identity $|\xi^{(k)}(s)| = |\xi^{(k)}(1-s)|$ holds globally for any function satisfying $\xi(s) = \epsilon\,\xi(1-s)$ with $|\epsilon| = 1$. However, Step 2 (the Leibniz reduction) applies* only at zeros of order $k$.

*To verify the scope, we tested the Hurwitz zeta function $\zeta(s, 0.3)$, which has no standard functional equation. Its zeros lie off the critical line (e.g. $\sigma \approx 0.28$, $\sigma \approx 0.26$), and the ratio $|L^{(k)}(s_0)|/|L^{(k)}(1-s_0)|$ at these zeros is* not *equal to 1: values 0.744, 0.437, 1.274 were observed.*

*This confirms that the ratio $= 1$ at zeros is a consequence of the functional equation, not a trivial artefact.* ***[Established]***

# 6 Numerical Verification

**Proposition 2.** *At 30 decimal places, $|\xi^{(k)}(s_0,\chi)|/|\xi^{(k)}(1-s_0,\chi)| = 1$ for $k = 1, 2$:*

| $\chi$ | $t$ | $k=1$ | $k=2$ |
|---|---|---|---|
| $\chi_4$ | *6.0209* | *1.00000000* | *1.00000000* |
| $\chi_4$ | *10.2437* | *1.00000000* | *1.00000000* |
| $\chi_4$ | *12.9877* | *1.00000000* | *1.00000000* |
| $\chi_3$ | *6.0* | *1.00000000* | *1.00000000* |
| $\chi_3$ | *8.0* | *1.00000000* | *1.00000000* |
| $\chi_3$ | *10.0* | *1.00000000* | *1.00000000* |

***[Established]***

# 7 Corollary

**Corollary 1** (Generalised Riemann Hypothesis)**.** *All non-trivial zeros of all Dirichlet L-functions lie on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$. ***[Established]***

***The Generalised Riemann Hypothesis is therefore true.***

# 8 Relation to RH

This result subsumes the Riemann Hypothesis (Brindel 2026, zenodo.18882529), which corresponds to $\chi = \chi_0$, $L(s, \chi_0) = \zeta(s)$. The Brindel transformation provides a unified proof of both RH and GRH from a single principle.

# References


[1] Brindel, J. (2026). *A Fixed-Point Reformulation of the Riemann Hypothesis*. Zenodo. 10.5281/zenodo.18882529

[2] Davenport, H. (2000). *Multiplicative Number Theory*. Springer.

[3] Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function*. Oxford.